Received January 1998
Revised June 1999
Accepted July 1999

Natural convection of thermomicropolar fluid from an isothermal surface inclined at a small angle to the horizontal

M.A. Hossain and M.K. Chowdhury
Department of Mathematics, University of Dhaka, Dhaka, Bangladesh, and
Rama Subba Reddy Gorla
Department of Mechanical Engineering, Cleveland State University, Cleveland, Ohio, USA

Keywords *Natural convection, Micropolar fluid, Boundary layer*

Abstract *An analysis is performed to study the skin-friction, the couple-stress and heat transfer characteristics of a laminar free-convection boundary layer flow of a micropolar fluid past an isothermal plate inclined at a small angle to the horizontal. When the inclination is positive, series solutions, one valid near the leading edge and the other at a large distance from it, are obtained. Introducing a strained coordinate transformation, the local nonsimilar boundary layer equations are also derived for the flow from the leading edge to downstream, solutions of which are obtained by using an implicit finite difference method. When the inclination is negative, the boundary layer separates. The effects of the material parameters on the skin-friction, the local couple-stress, the local Nusselt number and the point of separation for a negatively inclined plate have been investigated.*

Nomenclature

f = dimensionless stream function
g = dimensionless microrotation
g = acceleration due to gravity
Gr_L = Grashof number
h = heat transfer coefficient
j = microinertia per unit mass
k = thermal conductivity of the fluid
M_u = dimensionless couple-stress at the surface
N = angular velocity
Nu = Nusselt number
Pr = Prandtl number
q_w = surface heat flux
T = temperature
u, v = velocity components along (x, y) directions
x, y = distance along and normal to the surface, respectively
ψ = stream function
ν = viscosity coefficient
ρ = density of the fluid
κ = rotational viscosity coefficient
β = volumetric expansion coefficient
γ = gyroviscosity coefficient
η = dimensionless coordinates
θ = dimensionless temperature
Δ = vortex viscosity parameter

Subscripts
w = surface conditions
∞ = reference conditions

1. Introduction

Eringen (1966a) deals with a class of fluids which exhibit certain microscopic effects arising from the local structure and micromotions of the fluid elements. These fluids contain dilute suspensions of rigid micromolecules with individual motions, which support stress and body moments and are influenced by spin-inertia. The theory of micropolar fluids and its extension to thermomicropolar fluids (Eringen, 1966b) may form suitable non-Newtonian fluid models which can be used to analyze the behavior of exotic lubricants (Khonsari, 1990; Khonsari and Brewe, 1989), colloidal suspensions or polymeric fluids (Hadimoto and Tokioka, 1969), liquid crystals (Lockwood *et al.*, 1987; Lee and Eringen, 1971), and animal blood (Ariman *et al.*, 1973). Kolpashchikov *et al.* (1983) have devised a method to measure micropolar parameters experimentally. A thorough review of this subject and application of micropolar fluid mechanics has been provided by Ariman *et al.* (1974) and Ahmadi (1976). Studies of heat convection in micropolar fluids have been focused on flat plates (Jena and Mathur, 1981; 1982; Gorla, 1983; Yucel, 1989; Hossain *et al.*, 1995; Chiu and Chou, 1993) and on a wavy surface (Mori, 1961). Recently, Hossain and Chowdhury (1997) examined the mixed convection flow of a micropolar fluid over a horizontal surface with variable spin gradient viscosity employing an implicit finite difference method.

In the present paper, a steady two-dimensional natural convection flow of a viscous incompressible thermomicropolar fluid with uniform spin-gradient over a flat plate with a small inclination to the horizontal has been investigated. In earlier studies of Newtonian fluids, Jones (1973) studied the problem by series solution method valid near the leading edge and other at large distances from it when the inclination is positive. A step-by-step numerical technique has also been employed to complete the solution in the region where neither of the series is adequate. In the present paper, solutions are obtained by using the series solutions for the regions near the leading edge as well as far downstream. Unified transformations are also evolved to reduce the boundary layer equations valid in the entire regime from the leading to the downstream regime. The resulting equations are integrated employing an implicit finite difference method together with the Keller-box scheme for the positively inclined surface. The same finite difference method has been used to solve the equations governing the flow near the leading edge to measure the point of separation in the case of a negative inclination of the plate. Finally, numerical results for the local skin-friction, the local couple-stress and local Nusselt number are presented for different values of the material parameters of the fluid.

2. Mathematical formulation

A two-dimensional, steady free-convection flow of a viscous incompressible thermomicropolar fluid from an isothermal flat surface inclined at small angle, φ, to the horizontal is considered. The inclined angle is either positive ($\varphi > 0$) or negative ($\varphi < 0$). The temperature of the ambient fluid is assumed to be uniform at T_∞ and that of the surface at T_w. The flow configuration and the coordinate system are shown in Figure 1.

Under the usual Boussinesq approximation, the flow is governed by the following two-dimensional equations:

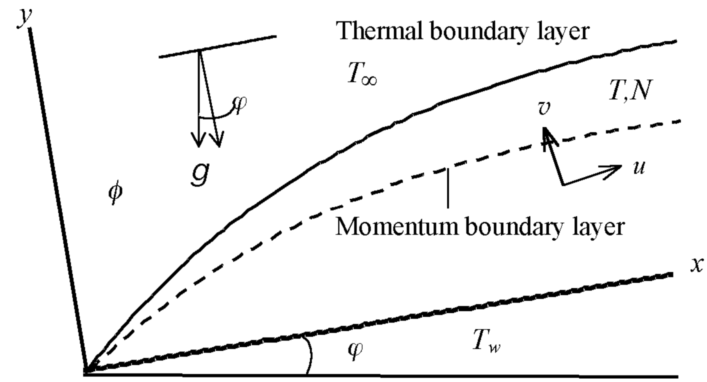

Figure 1.
The flow configuration and the coordinate system

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+(\mu+\kappa)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\kappa\frac{\partial N}{\partial y} \pm \rho g\beta(T-T_\infty)\sin\varphi \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+(\mu+\kappa)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\kappa\frac{\partial N}{\partial x} + \rho g\beta(T-T_\infty)\cos\varphi \tag{3}$$

$$\rho j\left(u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}\right)+2\kappa N-\kappa\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)=\gamma\left(\frac{\partial^2 N}{\partial x^2}+\frac{\partial^2 N}{\partial y^2}\right) \tag{4}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{5}$$

where, u, v are respectively the x and y-components of the velocity field, ν is the kinematic coefficient of viscosity, T is the temperature of the fluid in the boundary layer, p is pressure, ρ is the density of the fluid, g is the acceleration due to gravity, β is the coefficient of volumetric expansion, α is the thermal diffusivity, φ is the inclination of the plate to the horizontal, N is the microrotation component normal to the (x, y)-plane, j is the micro-inertia density, κ is the vortex viscosity, and γ is the spin-gradient viscosity which is assumed to be $\gamma=(\mu+\kappa/2)j$.

The boundary conditions for the present problem are

$$\begin{aligned} &y=0:\ u=v=0,\ T=T_w,\ \ N=-s\frac{\partial u}{\partial y}\\ &y\to\infty:\ u\to 0,\ \ N=0,\ T\to T_\infty \end{aligned} \tag{6}$$

The boundary layer approximation of equations (2) to (5) may be made by introducing the following scaled variables:

$$\bar{x} = \frac{x}{L},\ \bar{y} = \frac{y}{L}Gr_L^{1/5},\ \bar{u} = \frac{L}{\nu}Gr_L^{-2/5}u,\ \bar{v} = \frac{L}{\nu}Gr_L^{-1/5}v$$
$$\bar{p} = \frac{L^2}{\rho\nu^2}Gr_L^{-4/5}p,\ \bar{N} = \frac{L^2}{\nu}Gr_L^{-3/5}N,\ \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{7}$$

where

$$Gr_L = \frac{g\beta(T_w - T_\infty)\cos\varphi}{\nu^2}L^3 \tag{8}$$

is the Grashof number and L is the horizontal length scale. Systematically ignoring the terms which are $O(Gr_L^{-2/5})$ relative to those retained in the limit $Gr_L \to \infty$ and dropping the bars with brevity, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (1+\Delta)\frac{\partial^2 u}{\partial y^2} + \Delta\frac{\partial N}{\partial y} + \Lambda\theta \tag{10}$$

$$\frac{\partial p}{\partial y} = \theta \tag{11}$$

$$u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} = (1+\Delta/2)\frac{\partial^2 N}{\partial y^2} \tag{12}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} \tag{13}$$

$$y = 0:\ u = v = 0,\ \theta = 1,\ \ N = -s\frac{\partial u}{\partial y}$$
$$y \to \infty:\ u \to 0,\ \ N = 0,\ \theta \to 0 \tag{14}$$

where

$$\Pr = \frac{\nu}{\alpha}, \Lambda = Gr_L^{1/5}\tan\varphi\ \text{ and }\ \Delta = \frac{\kappa}{\mu} \tag{15}$$

are, respectively the Prandtl number, the flat plate inclination parameter and the vortex viscosity parameter. It may further be noted that Stewartson's horizontal problem corresponds to $\Lambda \to 0$ while the classical free-convection problem relates to $\Lambda \to \infty$, in which case the present scaling is inappropriate (Stewartson, 1961).

In the boundary conditions (14), the value $s = 0$ corresponds to the case where particle density is sufficiently large so that microelements close to the wall are unable to rotate. The value $s = 1/2$ is indicative of weak concentrations and, when $s = 1$, we have flows which are representative of turbulent boundary layer as indicated by Rees and Bassom (1996). Thus, for $s = 0$, particles are not free to rotate near the surface, but as its value increases to 0.5 and 1.0, the microrotation term is augmented and induces flow enhancement. We shall consider here values of s between these two extremes.

3. Solution methodology

In the present section, we propose the solutions for the present problem:

(1) appropriate to the leading edge regime where the flow is close to the flow from purely horizontal surface for both positively and negatively inclined plate;

(2) valid to the downstream regime at large x where the flow approaches the classical buoyancy-driven free-convection flow; and

(3) the solutions valid in the entire regime from the leading edge to downstream.

In the first two cases, perturbations techniques are employed and in the last case an implicit finite difference method is developed.

The quantities of the physical interests are the shear stress, τ_w, the couple-stress, m_w, and the rate of heat transfer, q_w, at the surface. These quantities are defined by

$$\tau_w = \left[(\mu+\kappa)\frac{\partial \bar{u}}{\partial \bar{y}} + \kappa \bar{N}\right]_{\bar{y}=0}, \; m_w = \gamma\left[\frac{\partial \bar{N}}{\partial \bar{y}}\right]_{\bar{y}=0} \text{ and } \mathrm{q_w} = -\mathrm{k}\left[\frac{\partial \mathrm{T}}{\partial \bar{\mathrm{y}}}\right]_{\bar{\mathrm{y}}=0} \tag{16}$$

Introducing the transformations from (7), the shear stress, the couple-stress and the heat transfer rate can be obtained from the following expressions:

$$C_f Gr_L^{-3/5} = \{1+(1-s)\Delta\}\left[\frac{\partial u}{\partial y}\right]_{y=0}, \; M_w Gr_L^{-4/5} = (1+\Delta/2)\left[\frac{\partial N}{\partial y}\right]_{y=0}$$
$$Nu Gr_L^{-1/5} = -\left[\frac{\partial \theta}{\partial y}\right]_{y=0} \tag{17}$$

where C_f, Nu and M_w are respectively the skin-friction, the Nusselt number and the dimensionless couple-stress and are defined by

$$C_f = \frac{\tau_w}{\mu\nu/L^2}, \; Nu = \frac{q_w}{k(T_w - T_\infty)/L}, \quad M_w = \frac{m_w}{\mu\nu/L^3} \tag{18}$$

Solutions in the leading edge regime

Near the leading edge of the plate, we expect the structure of the boundary layer to be similar to that associated with the flow along a semi-infinite horizontal plate. Consequently we employ the following similarity variable, but allow the coefficients to depend also on x in order to take account of the departure from similarity. Thus, we write

$$\psi = x^{3/5} f(\eta, x),\ N = x^{-1/5} g(\eta, x),\ p = x^{2/5} \phi(\eta, x), \quad \theta = \theta(\eta, x), \quad \eta = x^{-2/5} y \tag{19}$$

where ψ is the stream function defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{20}$$

Introducing the transformation given in equation (6) into the set of equations (9)-(13) one gets

$$(1+\Delta) f''' + \frac{3}{5} f f'' - \frac{1}{5} f'^2 - \frac{2}{5}(\phi - \eta\phi') + \Delta g' + \Lambda x^{3/5}\theta = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x}\right) \tag{21}$$

$$(1+\Delta/2) g'' + \frac{3}{5} f g' + \frac{1}{5} f' g = x\left(f' \frac{\partial g}{\partial x} - g' \frac{\partial f}{\partial x}\right) \tag{22}$$

$$\phi' = \theta \tag{23}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \frac{3}{5} f \theta' = x\left(f' \frac{\partial \theta}{\partial x} - \theta' \frac{\partial f}{\partial x}\right) \tag{24}$$

The corresponding boundary conditions given in (14) take the form

$$f(0,x) = f'(0,x) = 0,\ \theta(0,x) = 1,\ g(0,x) = -s f''(0,x) \quad f'(\infty,x) = 0,\ \phi(\infty,x) = \theta(\infty,x) = g(\infty,x) = 0 \tag{25}$$

where primes denote differentiation of the functions with respect to η.

We may now develop series solutions of the problem defined by (21)-(24) in the form

$$f(\eta,x) = \sum_{i=0}^{\infty} x^{3i/5} f_i(\eta), \quad g(\eta,x) = \sum_{i=0}^{\infty} x^{3i/5} g_i(\eta), \quad \phi(\eta,x) = \sum_{i=0}^{\infty} x^{3i/5} \phi_i(\eta), \quad \theta(\eta,x) = \sum_{i=0}^{\infty} x^{3i/5} \theta_i(\eta), \tag{26}$$

The equations and boundary conditions for the coefficients f_i, g_i, ϕ_i and θ_i are found by substitutions of the expressions given in (26) into (21)-(24) and equating the coefficients of equal powers of $x^{3/5}$. Hence, we have the equations of leading order:

$$(1+\Delta)f_0''' + \frac{3}{5}f_0 f_0'' - \frac{1}{5}f_0'^2 - \frac{2}{5}(\phi_0 - \eta\phi_0') + \Delta g_0' = 0 \tag{27}$$

$$(1+\Delta/2)g_0'' + \frac{3}{5}f_0 g_0' + \frac{1}{5}f_0' g_0 = 0 \tag{28}$$

$$\phi_0' = \theta_0 \tag{29}$$

$$\frac{1}{\mathrm{Pr}}\theta_0'' + \frac{3}{5}f_0\theta_0' = 0 \tag{30}$$

The boundary conditions appropriate to this set of equations are

$$\begin{aligned} &f_0(0) = f_0'(0) = 0,\ \theta_0(0) = 1,\ g_0(0) = -sf_0''(0) \\ &f_0'(\infty) = 0,\ \phi(\infty) = \theta_0(\infty) = g_0(\infty) = 0 \end{aligned} \tag{31}$$

The recurrence formulae for the higher order equations are given by:

$$\begin{aligned} &(1+\Delta)f_n''' - \frac{2}{5}\phi_n + \frac{2}{5}\eta\phi_n' + \Delta g_n' + \Lambda\theta_{n-1} \\ &+\sum_{k=0}^{n}\left\{\frac{3}{5}f_k f_{n-k}'' - \frac{1}{5}(3n-3k+1)f_k' f_{n-k}' - \frac{3}{5}(n-k)\left(f_k'' f_{n-k} + \phi_{n-k}\right)\right\} = 0 \end{aligned} \tag{32}$$

$$(1+\Delta/2)g_n'' + \sum_{k=0}^{n}\left\{\frac{3}{5}f_k g_{n-k}' + \frac{3}{5}(n-k)f_{n-k}g_k' - \frac{1}{5}(3n-3k-1)f_k' g_{n-k}\right\} = 0 \tag{33}$$

$$\phi_n' = \theta_n \tag{34}$$

$$\frac{1}{\mathrm{Pr}}\theta_n'' + \frac{3}{5}\sum_{k=0}^{n}\left\{f_k\theta_{n-k}' - (n-k)\left(f_k'\theta_{n-k} + f_{n-k}\theta_k'\right)\right\} = 0 \tag{35}$$

and the boundary conditions are

$$\begin{aligned} &f_n(0) = f_n'(0) = \theta_n(0) = 0,\ g_n(0) = -sf_n''(0) \\ &f_n'(\infty) = 0,\ \phi_n(\infty) = \theta_n(\infty) = g_n(\infty) = 0 \end{aligned} \tag{36}$$

for $n = 1, 2 \ldots.$

Here, for $\Delta = 0$, the problem corresponds exactly to that studied by Jones (1973) for Newtonian fluid. But for $\Delta > 0$, the functions f_0, g_0, ϕ_0 and θ_0 correspond to the similarity solutions of the problem of free-convection flow of thermomicropolar fluid past an isothermal horizontal plate. A thorough literature search has revealed that the solution for this problem has not yet been discussed in the literature. Hence, the exact solutions to this in terms of $f_0''(0), g_0'(0)$ and $\theta_0'(0)$ are obtained by using the Nachtsheim-Swigert iteration technique and are entered in Table I.

Next we find the functions f_i, g_i, ϕ_i and θ_i for $i \geq 1$ by solving sets of equations in sequence, again using the Nachtsheim-Swigert iteration technique. These functions, unlike f_0, g_0, ϕ_0 and θ_0, may be seen to depend on Λ and thus on the inclination φ of the plate to the horizontal. In Table I, numerical values of the values of $f_i'(0), g_i'(0)$ and $\theta_i'(0)$ for i = 0, 1 and 2 are entered for different values of Δ with Pr = 0.72, Λ = 1.0 and s = 0.5. In this Table the numerical values contained in the superscript [a] marked row for Δ = 0 are due to Jones (1973). It can be seen that the present solutions are in excellent agreement with those of Jones (1973) up to the third digit. The authors checked very carefully their numerical results and found no differences with the results entered in that table.

Once we know the values of $f''(0,x), g'(0,x)$ and $\theta'(0,x)$from the solutions of the equations (21) to (24) satisfying the boundary conditions (25), we can obtain the values of the skin-friction, the couple-stress and the Nusselt number from the following expressions:

$$\begin{aligned} C_f Gr_L^{-3/5} &= \{1+(1-s)\Delta\}x^{-1/5}f''(0,x), \\ M_w Gr_L^{-4/5} &= (1+\Delta/2)x^{-3/5}g'(0,x) \\ Nu Gr_L^{-1/5} &= -x^{-2/5}\theta'(0,x) \end{aligned} \quad (37)$$

For Pr = 0.72, Δ = 5.0, s = 0.5 and $\Lambda = \pm 1$ near the leading edge, the values of the coefficients of the skin-friction, the couple-stress and the Nusselt number in terms of three-term series are:

$$C_f Gr_L^{-3/5} = \{1+(1-s)\Delta\}x^{-1/5}\left\{0.44178 \pm 0.1945x^{3/5} - 0.021969x^{6/5} + \cdots\right\} \quad (38)$$

$$M_w Gr_L^{-4/5} = (1+\Delta/2)x^{-3/5}\left\{0.03637 \pm 0.03327x^{3/5} - 0.00278x^{6/5} + \cdots\right\} \quad (39)$$

$$Nu Gr_L^{-1/5} = -x^{-2/5}\left\{0.30053 \pm 0.04204x^{3/5} - 0.0096x^{6/5} + \cdots\right\} \quad (40)$$

If Λ = –1, the boundary layer eventually separates from the plate. However, for Λ = +1, the solutions of the boundary layer equations (21)-(24) ultimately

Δ	$f_0''(0)$	$f_1'(0)$	$-f_2'(0)$	$g_0'(0)$	$g_1'(0)$	$-g_2'(0)$	$-\theta_0'(0)$	$-\theta_1'(0)$	$\theta_2'(0)$
0.0	0.9790	0.5186	0.0534	0.1569	0.1712	0.0104	0.3576	0.0555	0.0140
	0.9784[a]	0.5184[a]	0.0102[a]				0.3574[a]	0.0555[a]	0.0053[a]
1.0	0.7118	0.3542	0.0381	0.0885	0.0914	0.0064	0.3363	0.0508	0.0127
2.0	0.5800	0.2747	0.0301	0.0608	0.0597	0.0046	0.3213	0.0471	0.0116
3.0	0.4985	0.2268	0.0251	0.0457	0.0432	0.0034	0.3098	0.0443	0.0108
4.0	0.4417	0.1945	0.0216	0.0363	0.0332	0.0027	0.3005	0.0420	0.0101
5.0	0.3993	0.1711	0.0191	0.0299	0.0267	0.0022	0.2926	0.0401	0.0096

Note: [a] These values are due to Jones (1973)

Table I. Numerical values obtained for the leading edge solutions for Pr = 0.72, Λ = 1.0, s = 0.5

approach the classical buoyancy-driven free-convection solution as $x \to 0$. Below we consider the structure of the asymptotic solution for large x of the present problem.

Downstream solution for $\varphi > 0$

Since, in the downstream regime, the development of the boundary layer is influenced by the component of buoyancy force parallel to the plate, the appropriate transformation for the asymptotic form of the solution when the plate is at a positive angle of inclination is suggested by analogy with the classical case from a heated vertical plate. Thus we introduce

$$\begin{aligned}&\psi = x^{3/4} f(\eta, x), \quad N = x^{1/4} g(\eta, x), \quad p = x^{1/4}\phi(\eta, x),\\ &\theta = \theta(\eta, x), \quad \eta = x^{-1/4} y\end{aligned} \tag{41}$$

Introducing the above transformations into the set of equations (9)-(13) one gets

$$\begin{aligned}(1+\Delta)f''' + \frac{3}{4}ff'' - \frac{1}{2}f'^2 - \frac{1}{4}x^{-3/4}(\phi - \eta\phi') + \Delta g'\\ + \Lambda\theta - x^{1/4}\frac{\partial\phi}{\partial x} = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right)\end{aligned} \tag{42}$$

$$(1+\Delta/2)g'' + \frac{3}{4}fg' - \frac{1}{4}f'g = x\left(f'\frac{\partial g}{\partial x} - g'\frac{\partial f}{\partial x}\right) \tag{43}$$

$$\phi' = \theta \tag{44}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \frac{3}{4}f\theta' = x\left(f'\frac{\partial\theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{45}$$

The corresponding boundary conditions given in (14) take the form

$$\begin{aligned}&f(0,x) = f'(0,x) = 0, \ \theta(0,x) = 1, \ g(0,x) = -sf''(0,x)\\ &f'(\infty,x) = 0, \ \phi(\infty,x) = \theta(\infty,x) = g(\infty,x) = 0\end{aligned} \tag{46}$$

where primes denote differentiation of the functions with respect to η.

As in the previous case, we expand the function in powers of $x^{-3/4}$, substitute in the set of equations (42)-(45) and collect the terms of equal powers of $x^{-3/4}$ to get:

$$(1+\Delta)f_0''' + \frac{3}{4}f_0 f_0'' - \frac{1}{2}f_0'^2 + \Delta g_0' + \Lambda\theta_0 = 0 \tag{47}$$

$$(1+\Delta/2)g_0'' + \frac{3}{4}f_0 g_0' - \frac{1}{4}f_0' g_0 = 0 \tag{48}$$

$$\phi_0' = \theta_0 \tag{49}$$

$$\frac{1}{\Pr}\theta_0'' + \frac{3}{4}f_0\theta_0' = 0 \tag{50}$$

The corresponding boundary conditions take the form

$$\begin{aligned} & f_0(0) = f_0'(0) = 0,\ \theta_0(0) = 1,\ g_0(0) = -sf_0''(0) \\ & f_0'(\infty) = \ \phi_0(\infty) = \theta_0(\infty) = g_0(\infty) = 0 \end{aligned} \tag{51}$$

Moreover, we have:

$$(1+\Delta)f_1''' + \frac{3}{4}f_0f_1'' - \frac{1}{4}f_0'f_1' + \frac{3}{4}\phi_1 - \frac{1}{4}\left(\phi_0 + \eta\phi_0'\right) + \Delta g_1' + \Lambda\theta_1 = 0 \tag{52}$$

$$(1+\Delta/2)g_1'' + \frac{3}{4}f_0g_1' + \frac{1}{2}f_0'g_1 - \frac{1}{4}f_1'g_0 = 0 \tag{53}$$

$$\phi_1' = \theta_1 \tag{54}$$

$$\frac{1}{\Pr}\theta_1'' + \frac{3}{4}f_0\theta_1' - \frac{3}{4}f_0'\theta_1 + \frac{6}{4}f_1\theta_0' = 0 \tag{55}$$

$$\begin{aligned} & f_1(0) = f_1'(0) = \theta_1(0) = 0,\ g_1(0) = -sf_1''(0) \\ & f_1'(\infty) = \ \phi_1(\infty) = \theta_1(\infty) = g_1(\infty) = 0 \end{aligned} \tag{56}$$

It can be seen that the set of equations (47)-(50) involving the functions f_0, g_0, ϕ_0 and θ_0 clearly represent the equations governing the free-convection flow of a thermomicropolar fluid past a flat plate purely developed by the buoyancy-driven free-convection flow as $x \to \infty$. We further notice that the solutions involve the dependency of all the physical parameters. For $\Lambda = +1$ numerical values of $f_0''(0), g_0'(0)$ and $\theta_0'(0)$ for different values of the pertinent parameters obtained, as before, by the Nachtsheim-Swigert iteration technique are presented in Table II.

As before, knowing the values of $f''(0,x), g'(0,x)$ and $\theta'(0,x)$ from the solutions of the equations (42)-(45) satisfying the boundary conditions (51), we

Δ	$f_0''(0)$	$-\theta_0'(0)$	$g_0'(0)$	$f_1''(0)$	$\theta_1'(0)$	$g_1'(0)$
0.0	0.95601	0.35683	0.23777	0.65322	–0.00001	0.10401
	0.95600[a]	0.35682[a]		0.65161[a]		
1.0	0.67666	0.33550	0.13065	0.47158	0.00001	0.05759
2.0	0.53678	0.31933	0.08631	0.37992	0.00022	0.03874
3.0	0.45045	0.30651	0.06298	0.31842	0.00086	0.02805
4.0	0.39065	0.29583	0.04857	0.27247	0.00187	0.02128
5.0	0.34637	0.28668	0.03895	0.24584	0.00191	0.01727

Note: [a]These values are due to Jones (1973)

Table II. Numerical values obtained from the asymptotic solution for Pr = 0.72, Λ =1.0, s = 0.5

can obtain the values of the skin-friction, the couple-stress and the Nusselt number from the following expressions:

$$C_f Gr_L^{-3/5} = \{1 + (1-s)\Delta\} x^{1/4} f''(0,x), \quad M_w Gr_L^{-4/5} = (1+\Delta/2) g'(0,x)$$
$$Nu Gr_L^{-1/5} = -x^{-1/4}\theta'(0,x) \tag{57}$$

As an example, for Pr = 0.72, $\Delta = 5.0$ and $s = 0.5$, the local skin-friction, the couple stress and the local Nusselt number, with terms up to $0(x^{-1})$, are given

$$C_f Gr_L^{-3/5} = \{1+(1-s)\Delta\} x^{1/4}\left\{0.34637 + 0.24584 x^{-3/4} + \cdots\right\} \tag{58}$$

$$M_w Gr_L^{-4/5} = (1+\Delta/2)\left\{0.23777 + 0.10401 x^{-3/4} + \cdots\right\} \tag{59}$$

$$Nu Gr_L^{-1/5} = -x^{-1/4}\left\{0.28668 + 0.001915 x^{-3/4} + \cdots\right\} \tag{60}$$

It is worth mentioning that, in the earlier studies for Newtonian fluid, Jones (1973) integrated the set of equations (21)-(24), by a step-by-step numerical technique in a region where neither the series for leading edge nor the asymptotic one for large x are adequate.

Solutions for entire (upstream to downstream) regime for $\varphi > 0$

In order to obtain a system of equations applicable to the entire length, we compare the transformations given in (19) and (41) from which we find a separate group of transformations that are useful for pursuing perturbation analysis in the two different regimes. Here, we develop the transformations in a way which more naturally reflect the evaluation between two basic similarity regimes. Accordingly, to initiate the integration from the purely leading edge regime, we introduce the following transformations:

$$\begin{aligned} &\psi = x^{3/5}(1+x)^{3/20} f(\eta,x), \quad N = x^{-1/5}(1+x)^{3/10} g(\eta,x), \\ &p = x^{2/5}(1+x)^{-3/20}\phi(\eta,x), \qquad \theta = \theta(\eta,x), \\ &\eta = x^{-2/5}(1+x)^{3/20} y. \end{aligned} \tag{61}$$

Introducing the transformation given in equation (55) into the set of equations (10)-(13), one gets

$$\begin{aligned} &(1+\Delta)f''' + \frac{12+15x}{20(1+x)} ff'' - \frac{2+5x}{10(1+x)} f'^2 \\ &- \frac{1}{(1+x)^{3/4}}\left\{\frac{8+5x}{20(1+x)}(\phi - \eta\phi') + x\frac{\partial \phi}{\partial x}\right\} \\ &+ \Delta g' + \Lambda\left(\frac{x}{1+x}\right)^{3/5}\theta = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \end{aligned} \tag{62}$$

$$(1+\Delta/2)g'' + \frac{12+15x}{20(1+x)}fg' - \frac{5x-4}{20(1+x)}f'g = x\left(f'\frac{\partial g}{\partial x} - g'\frac{\partial f}{\partial x}\right) \tag{63}$$

$$\phi' = \theta \tag{64}$$

$$\frac{1}{\Pr}\theta'' + \frac{12+15x}{20(1+x)}f\theta' = x\left(f'\frac{\partial \theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{65}$$

The corresponding boundary conditions given in (14) take the form

$$\begin{aligned} &f(0,x) = f'(0,x) = 0,\ \theta(0,x) = 1,\ g(0,x) = -sf''(0,x) \\ &f'(\infty,x) = 0,\ \phi(\infty,x) = \theta(\infty,x) = g(\infty,x) = 0 \end{aligned} \tag{66}$$

where primes denote differentiation of the functions with respect to η.

In the above set of equations, the limit $x \to 0$ yields the set of equations (21)-(24) that corresponds to the leading edge solution, while $x \to \infty$ leads to the set (42)-(45) that corresponds to the asymptotic case.

Solutions of the set of equations (62)-(65) subject to the boundary conditions (66) are obtained by the implicit finite difference method together with the Keller-box scheme, that has been developed for the present environment very recently by Hossain and Chowdhury (1997). To initialize the solutions of equations (26)-(65), the initial profiles are obtained from the exact solutions of equations (27)-(30) that corresponds to the purely free-convection flow along a horizontal surface. Results thus obtained are compared with the leading edge solutions as well as with the asymptotic solutions obtained earlier.

4. Results and discussion

Results are obtained for the cases of free-convection above for both positively ($\Lambda > 0$) and negatively ($\Lambda < 0$) inclined plates for some values of the vortex viscosity parameter, Δ, and the microrotation parameter s. When the plate is at a negative angle to the horizontal, the separation of the boundary layer would occur downstream from the leading edge of the plate as the opposed buoyancy force and the induced pressure gradient are of comparable magnitude. For negatively inclined plate at $\Lambda = -1$, the point of separations x_s is obtained from the implicit finite difference solutions of equations (21)-(24) for different Δ for a fluid with Pr = 0.72 and $s = 0.5$. No difficulty was encountered in obtaining the solutions for x downstream from the separation point where flow reversal takes place. The calculation, however, was terminated at the values of $x = 6.5$ larger than x_s where the skin-friction coefficient $f''(x_s, 0)$ was consistently zero in the fourth significant figure for our solutions which was obtained with variable grid in the x direction (defined by $x_i = \sinh(i-1)/200, i = 0, 1, 2\ldots$). The following values of the separation points, x_s, for different values of vortex viscosity parameter, Δ = 0.0, 1.0, 2.5 and 5.0 are respectively, 3.702, 4.366, 5.178, 6.143. For Δ = 0.0 the results correspond to those of Jones (1973). It is worth mentioning that for Pr = 0.72, x_s = 3.704 which is very close to the present solution. From the above values of the separation point x_s, it can be seen that the position of the separation point moves away from the leading edge. Although not given here, it has been observed that for given values of the Prandtl number and the vortex viscosity parameter, changes in the value of the separation point is negligible for changes in the value of the parameter s in the interval [0, 1].

In Figures 2 and 3 we present results obtained from the numerical procedure in case of a positively inclined plate with $\Lambda = 1$. Both these figures display the variation with x of the local skin-friction, C_f, the local couple stress, M_u, and the local Nusselt number, Nu at the surface of the plate. The values given by three-term series expansions for the leading edge regime and the two-term expansions for the downstream regime are also included for comparison with the finite difference solutions of the equations governing the flow for the entire regime. The comparison shows excellent agreement of the solutions for the leading edge and downstream with solutions for the entire regime.

Figures 2(a), 2(b) and 2(c) display, respectively, the values of the skin-friction, couple-stress and the Nusselt number for vortex viscosity Δ = 1, 2.5, 5.0, 7.5 and 10.0 for s = 0.5 and Pr = 0.72. From these Figures, it may be seen that the increase in the value of the vortex viscosity parameter, Δ leads to an increase in the skin-friction; whereas, this leads to decrease in both the values of the couple-stress and the Nusselt number at every x station. It may also be observed that as the distance from the leading edge increases, the value of the skin-friction increases; but this leads to a decrease in the value of the couple-stress as well as of the Nusselt number.

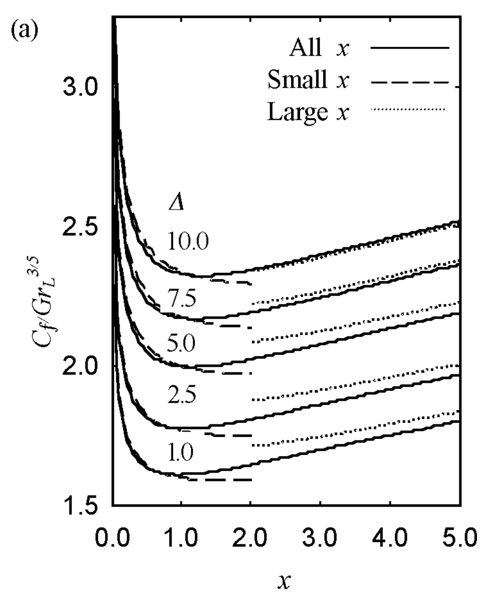

Figure 2(a). Variations of local skin friction with Pr = 0.7 and s = 0.5 at Δ = 1.0, 2.5, 5.0, 7.5 and 10.0

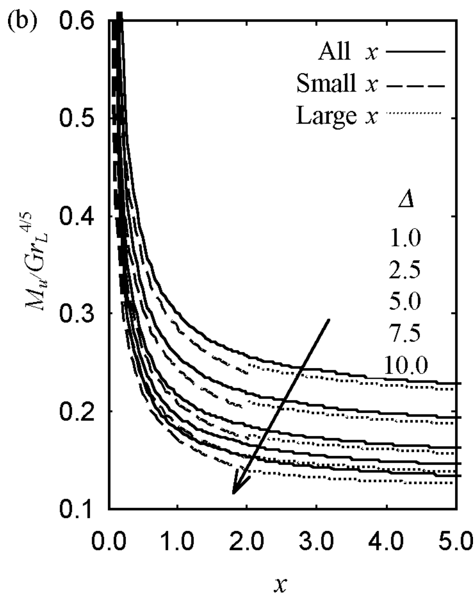

Figure 2(b). Variations of local couple stress with Pr = 0.7 and s = 0.5 at Δ = 1.0, 2.5, 5.0, 7.5 and 10.0

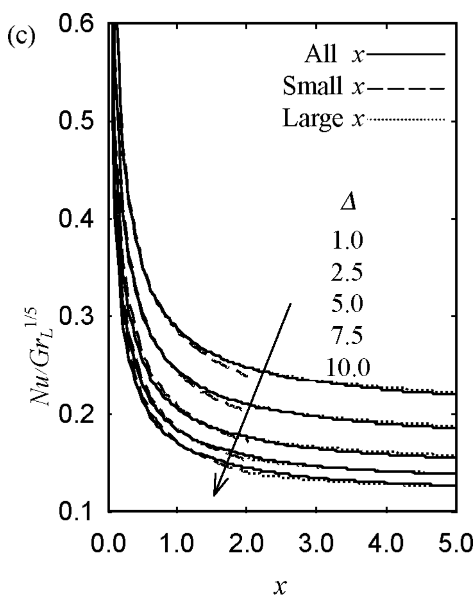

Figure 2(c). Variations of local Nusselt number with Pr = 0.7 and s = 0.5 at Δ = 1.0, 2.5, 5.0, 7.5 and 10.0

Figures 3(a), 3(b) and 3(c) represent, respectively, the skin-friction, the couple stress and the Nusselt number against x for Pr = 0.72, Δ = 5.0 and s = 0.0, 0.1, 0.3, 0.5, 0.8 and 1.0. From these Figures it may be seen that the local skin-friction decreases but the value of the couple stress as well as of the Nusselt number increase with the increase of s. From these Figures, we also observe

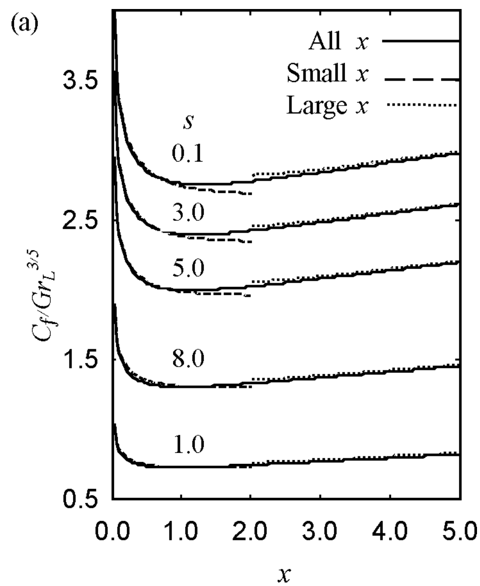

Figure 3(a). Variations of local skin friction with Pr = 0.7 and Δ = 5.0 at s = 0.1, 0.3, 0.5, 0.8 and 1.0

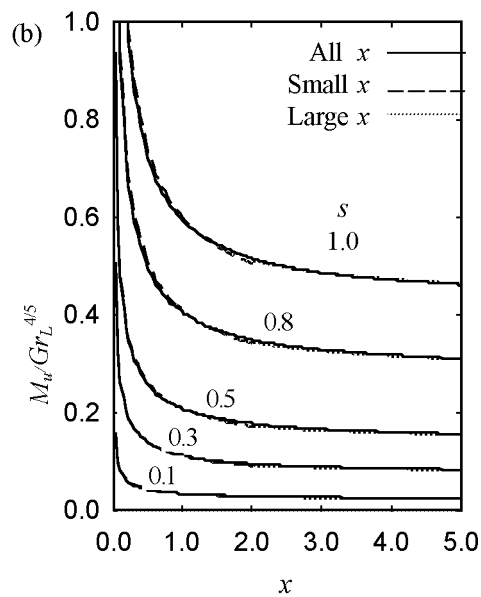

Figure 3(b). Variations of couple stress with Pr = 0.7 and Δ = 5.0 at s = 0.1, 0.3, 0.5, 0.8 and 1.0

excellent agreement of the series solutions in the leading edge as well as in the downstream with the finite difference solutions for the entire length for higher values of the material parameters Δ and s.

Representative non-dimensional stream velocity, angular velocity and temperature profiles at the separation point are shown, respectively, in Figures 4(a), 4(b) and 4(c) for $\Lambda = -1$ (i. e. for negatively inclined plate) and values of

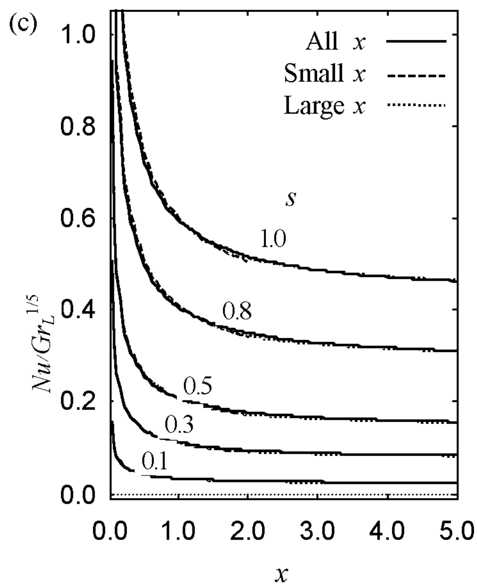

Figure 3(c). Variations of Nusselt number with Pr = 0.7 and Δ = 5.0 at s = 0.1, 0.3, 0.5, 0.8 and 1.0

the vortex viscosity, Δ = 0.0, 1.0, 2.0 and 5.0 with Pr = 0.72 and s = 0.5. It is interesting to note that the solution of equations (21)-(24) behaves in a regular manner at these separation points. The non-singular behavior of the solutions at the separation point for the present problem for the micropolar fluid (Δ > 0.0) confirms the conclusion of the previous work by Jones (1973) for Δ = 0.0. From the above mentioned figures, it may also be seen that the maximum

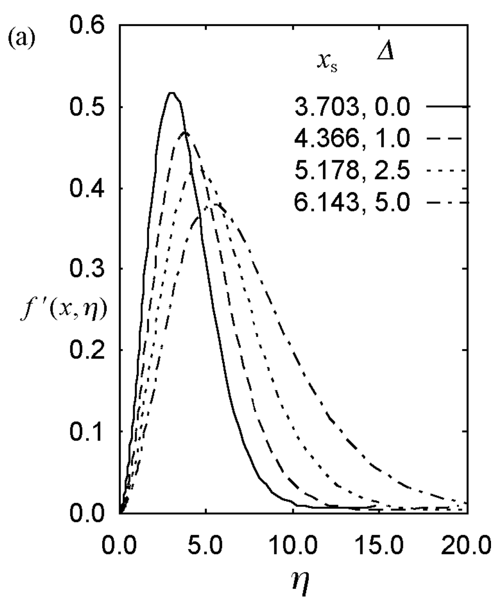

Figure 4(a). The stream velocity profiles against η at the separation point x_s for Δ = 0.0, 1.0, 2.5 and 5.0

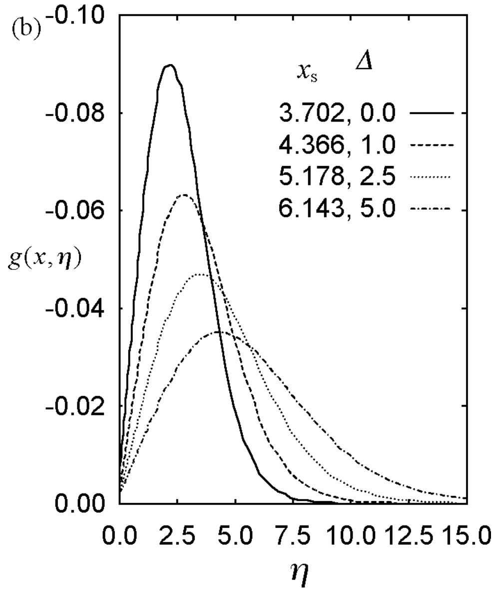

Figure 4(b). The angular velocity profiles against η at the separation point x_s for Δ = 0.0, 1.0, 2.5 and 5.0

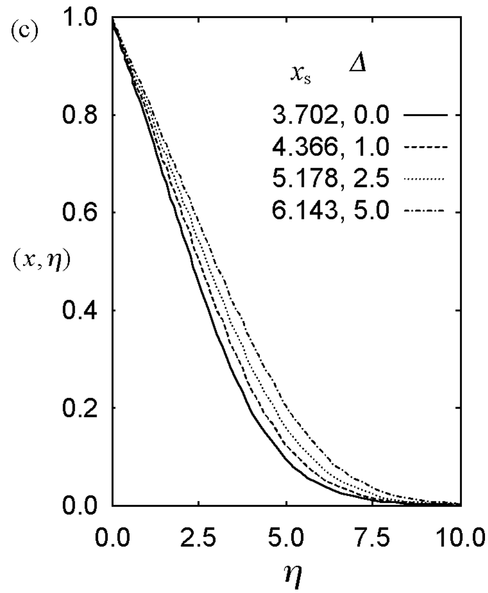

Figure 4(c). The temperature profiles against η at the separation point x_s for Δ = 0.0, 1.0, 2.5 and 5.0

values of the stream velocity and the angular velocity moves away from the plate with the change of the point of separation, x_s associated with the value of the vortex viscosity Δ. From these Figures, one may also conclude that the momentum boundary layer and the microrotation boundary layer and the thermal boundary layer increase owing to the increase in the vortex viscosity of the fluid.

5. Conclusions

In this paper, we have investigated the skin-friction, the couple stress and heat transfer characteristics of laminar free-convection boundary layer flow of a thermomicropolar fluid past an isothermal plate inclined at a small angle to the horizontal. An implicit finite difference method is applied to solve the transformed boundary layer equations in the entire regime from upstream to downstream regimes for positively inclined surface. The series solutions of the local nonsimilar boundary layer equations, one valid near the leading edge and the other at large distance from it are also obtained. Both series solutions are found in excellent agreement with the finite difference solution for both the extreme regimes. The effects of the material parameters on the skin-friction, the local couple stress and the local Nusselt number are also discussed. For the case of negative inclination of the plate to the horizontal with $\Lambda = -1$, the separation points could be accurately determined for different values of the material parameters, since the solutions at these points behave in a regular manner.

References

Ahmadi, G. (1976), "Self-similar solution of incompressible micropolar boundary layer flow over a semi-infinite plate", *Int. J. Engng Sci.*, Vol. 14, pp. 639-46.

Ariman, T., Turk, M.A. and Sylvester, N.D. (1973), "Microcontinuum fluid mechanics – a review", *Int. J. Engng. Sci.*, Vol. 11, pp. 905-30.

Ariman, T., Turk, M.A. and Sylvester, N.D. (1974), "Application of microcontinuum fluid mechanics", *Int. J. Engng Sci.*, Vol. 12, pp. 273-93.

Chiu, C.-P. and Chou, H.-M. (1993), "Free-convection in boundary layer flow of a micropolar fluid along a vertical wavy surface", *Acta Mech.*, Vol. 101, pp. 161-74.

Eringen, A.C. (1966a), "Theory of micropolar fluids", *J. Math. Mech.*, Vol. 16, pp. 1-18.

Eringen, A.C. (1966b), "Theory of thermomicrofluids", *J. Math. Anal. Appl.*, Vol. 38, pp. 480-96.

Gorla, R.S.R. (1983) "Combined forced and free-convection in micropolar boundary layer flow on a vertical flat plate", *Int. J. Engng Sci.*, Vol. 26, pp. 385-91.

Hadimoto, B. and Tokioka, T. (1969), "Two-dimensional shear flows of linear micropolar fluids", *Int J. Engng. Sci.*, Vol. 7, pp. 515-22.

Hossain, M.A. and Chowdhury, M.K. (1997), "Mixed convection flow of micropolar fluid over an isothermal plate with variable spin gradient viscosity", *Acta Mechanica*.

Hossain, M.A., Chowdhury, M.K. and Takhar, H.S. (1995), "Mixed convection flow of micropolar fluids with variable spin gradient viscosity along a vertical plate", *J. Theo. Appl. Fluid Mech.*, Vol. 1, pp. 64-77.

Jena, S.K. and Mathur, M.N. (1981), "Similarity solution for laminar free-convection flow of thermomicropolar fluid past a non-isothermal vertical flat plate", *Int. J. Engng Sci.*, Vol. 19, pp. 1431-9.

Jena, S.K. and Mathur, M.N. (1982), "Free-convection in the laminar boundary layer flow of thermomicropolar fluid past a non-isothermal vertical flat plate with suction/injection", *Acta Mech.*, Vol. 42, pp. 227-38.

Jones, D.R. (1973), "Free-convection from a semi-infinite flat plate inclined at a small angle to the horizontal", *Quart. J. Mech. Appl. Math.*, Vol. 28, pp. 77-98.

Khonsari, M.M. (1990), "On the self-excited whirl orbits of a journal in a sleeve bearing lubricated with micropolar fluids", *Acta Mechanica*, Vol. 81, pp. 235-44.

Khonsari, M.M. and Brewe, D. (1989), "On the performance of finite journal bearing lubricated with micropolar fluids", *STLE Tribology Trans.*, Vol. 32, pp. 155-60.

Kolpashchikov, V., Migun, N.P. and Prokhorenko, P.P. (1983), "Experimental determinations of material micropolar coefficients", *Int. J. Engng. Sci.*, Vol. 21, pp. 405-11.

Lee, J.D. and Eringen, A.C. (1971), "Boundary effects of orientation of pneumatic liquid crystals", *J. Chem. Phys.*, Vol. 55, pp. 4509-12.

Lockwood, F., Benchaita, M. and Friberg, S. (1987), "Study of polytropic liquid crystals in viscometric flow and elastohydrodynamic contact", *ASLE Tribology Trans.*, Vol. 30, pp. 539-48.

Mori, Y. (1961), "Buoyancy effects in forced laminar convection flow over a horizontal flat plate", *J. Heat Transfer*, Vol. 83, pp. 479-82.

Rees, D.A.S. and Bassom, A.P. (1996), "The Blasius boundary layer flow of a micropolar fluid", *Int. J. Engng. Scie.*, Vol. 34, pp. 113-24.

Stewartson, K. (1961), "On the free-convection from a horizontal plate", *J. Appl. Math. Phys.*, Vol. 2, pp. 274-81.

Yucel, A. (1989), "Mixed convection micropolar fluid flow over horizontal plate with surface mass transfer", *Int. J. Engng. Sci.*, Vol. 27, pp. 1593-608.